Mixed-Integer Vertex Covers on Bipartite Graphs

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Abstract. Let A be the edge-node incidence matrix of a bipartite graph $G = (U, V; E)$, I be a subset of the nodes of G, and b be a vector such that $2b$ is integral. We consider the following mixed-integer set:

$$X(G, b, I) = \{x : Ax \geq b,\ x \geq 0,\ x_i \text{ integer for all } i \in I\}.$$

We characterize $\text{conv}(X(G, b, I))$ in its original space. That is, we describe a matrix (C, d) such that $\text{conv}(X(G, b, I)) = \{x : Cx \geq d\}$. This is accomplished by computing the projection onto the space of the x-variables of an extended formulation, given in [1], for $\text{conv}(X(G, b, I))$. We then give a polynomial-time algorithm for the separation problem for $\text{conv}(X(G, b, I))$, thus showing that the problem of optimizing a linear function over the set $X(G, b, I)$ is solvable in polynomial time.

1 Introduction

Given a bipartite graph $G = (U, V; E)$, a vector $b = (b_e)_{e \in E}$, with the property that b is *half-integral*, i.e. $2b_e \in \mathbb{Z}$, $e \in E$, and a set $I \subseteq (U \cup V)$, we consider the problem of characterizing the convex hull of all nonnegative $x \in \mathbb{R}^{U \cup V}$ such that

$$\begin{aligned} x_i + x_j &\geq b_{ij} \quad \text{for every } ij \in E, \\ x_i &\in \mathbb{Z} \quad \text{for every } i \in I. \end{aligned}$$

That is, given the edge-node incidence matrix A of a bipartite graph G, a partition (I, L) of its column-set, and an half-integral vector b, we consider the following mixed-integer set:

$$X(G, b, I) = \{x : Ax \geq b,\ x \geq 0,\ x_i \text{ integer for all } i \in I\}. \tag{1}$$

In this paper we provide a *formulation* for the polyhedron $\text{conv}(X(G, b, I))$, where a formulation for a polyhedron P is a description of P as the intersection of a finite number of half-spaces. So it consists of a finite set of inequalities $Cx \geq d$ such that $P = \{x : Cx \geq d\}$.

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An *extended formulation* of P is a formulation for a polyhedron P' in a higher dimensional space that includes the original space, so that P is the projection of P' onto the original space.

A general technique to describe an extended formulation for the set of solutions of a system $Ax \geq b$, when $A^\top$ is a network matrix and some of the variables are restricted to be integer, was introduced in [1]. In Section 2 we derive such an extended formulation for $\mathrm{conv}(X(G, b, I))$, while in Section 3 we describe a formulation in the original space by explicitly computing the projection of the polyhedron defined by the extended formulation. Finally, in Section 4, we give a polynomial-time algorithm to solve the separation problem for $\mathrm{conv}(X(G, b, I))$.

1.1 The Main Result

Given a bipartite graph $G = (U, V; E)$, a partition (I, L) of $U \cup V$, and an half-integral vector b, we say that a path P of G is an *I-path* if at least one endnode of P is in I, and no intermediate node of P is in I. We say that P is *odd* if P has an odd number of edges e such that $b_e = \frac{1}{2} \bmod 1$. Whenever we have a vector v with entries indexed by some set S, given a subset T of S we denote $v(T) = \sum_{i \in T} v_i$. In this paper we show the following:

Theorem 1. *The polyhedron* $\mathrm{conv}(X(G, b, I))$ *is defined by the following inequalities:*

$$x_i + x_j \geq b_{ij} \qquad ij \in E, \tag{2}$$
$$2x(V(P) \cap L) + x(V(P) \cap I) \geq b(P) + \tfrac{1}{2} \qquad P \text{ odd } I\text{-path}, \tag{3}$$
$$x_i \geq 0 \qquad i \in U \cup V. \tag{4}$$

Eisenbrand [4] conjectured that the inequalities in (2)-(4) are sufficient to characterize $\mathrm{conv}(X(G, b, I))$ when G is a path. Theorem 1 shows that this conjecture holds in a quite more general setting (and it certainly cannot be extended beyond that). Preliminary results for the path case were obtained by Skutella [11] and Eisenbrand [4].

1.2 First Chvátal Closure

The following observation allows us to describe $X(G, b, I)$ in terms of a pure integer set.

Observation 2. *Let $\bar{x}$ be a vertex of* $\mathrm{conv}(X(G, b, I))$. *Then $2\bar{x}$ is integral.*

Proof: If not, let U' and V' be the sets of nodes i in U and V, respectively, such that $2\bar{x}_i$ is not integer. Then, for ϵ small enough, the vectors $\bar{x} + \epsilon\chi^{U'} - \epsilon\chi^{V'}$ and $\bar{x} - \epsilon\chi^{U'} + \epsilon\chi^{V'}$ are both in $\mathrm{conv}(X(G, b, I))$, where we denote by χ^S the incidence vector of S for any $S \subseteq U \cup V$. □

Let $b' = 2b$, A' be obtained form A by multiplying by 2 the columns corresponding to nodes in I. By Observation 2, the linear transformation $x'_i = x_i, i \in I$,

$x'_i = 2x_i, i \in L$, maps $X(G, b, I)$ into $\{x' : A'x' \geq b', x' \geq 0, x' \text{ integral}\}$, which is a pure integer set.

Let $P = v_1, \dots v_n$ be an I-path. Notice that $b(P) = \frac{1}{2} \bmod 1$ is equivalent to $b'(P)$ odd. Then the inequality

$$\sum_{i \in V(P)} x'_i \geq \left\lceil \frac{b'(P)}{2} \right\rceil \tag{5}$$

is a Gomory-Chvátal inequality of $\{x' : A'x' \geq b', x' \geq 0\}$. Indeed, assume $v_1 \in I$. If $v_n \in I$, then (5) is obtained from

$$\frac{1}{2}(2x'_{v_1} + x'_{v_2} \geq b'_{v_1 v_2}) + \sum_{i=2}^{n-2} \frac{1}{2}(x'_{v_i} + x'_{v_{i+1}} \geq b'_{v_i v_{i+1}}) + \frac{1}{2}(x'_{v_{n-1}} + 2x'_{v_n} \geq b'_{v_{n-1} v_n})$$

by rounding up the right-hand-side. If $x_n \notin I$, then (5) is obtained from

$$\frac{1}{2}(2x'_{v_1} + x'_{v_2} \geq b'_{v_1 v_2}) + \sum_{i=2}^{n-1} \frac{1}{2}(x'_{v_i} + x'_{v_{i+1}} \geq b'_{v_i v_{i+1}}) + \frac{1}{2}(x'_{v_n} \geq 0)$$

by rounding up the right-hand-side.

Furthermore the inequalities in (5) correspond to the inequalities in (3). Therefore Theorem 1 implies that the polyhedron defined by $A'x' \geq b'$, $x' \geq 0$ has Chvátal rank 1. In the case where G is a path with no intermediate node in I, this last fact follows immediately from a theorem of Edmonds and Jonhnson [2,3], since in this case A' satisfies the condition that the sum of the absolute values of the entries of each column is at most 2.

1.3 The Motivation

A (general) mixed-integer set is a set of the form

$$\{x \mid Ax \geq b, \ x_i \text{ integer } i \in I\} \tag{6}$$

where I is a subset of the columns of A and b is a vector that may contain fractional components.

In [1], it is shown that the problem of deciding if the above set is nonempty is NP-complete, even if b is an half-integral vector and A is a network matrix. (We refer the reader to [7] or [10] for definitions and results related to network matrices and, more generally, totally unimodular matrices.)

However, it may be possible that, when A is the *transpose* of a network matrix, the associated mixed-integer programming problem is polynomially solvable. Indeed, let MIX^{2TU} be a mixed-integer set of the form (6) when $A^\top$ is a network matrix.

An extended formulation of the polyhedron $\text{conv}(MIX^{2TU})$ was described in [1]. The extended formulation involves an additional variable for each possible fractional part taken by the variables at any vertex of $\text{conv}(MIX^{2TU})$. If this number is polynomial in the size of (A, b), then such a formulation is compact, i.e. of polynomial size in the size of (A, b). Therefore the problem of optimizing a linear function over MIX^{2TU} can be efficiently solved in this case. However, it seems to be rather difficult to compute the projection in the original x-space. It follows from Observation 2 that if $\bar{x}$ is a vertex of $\text{conv}(X(G, b, I))$, then $\bar{x}_i - \lfloor \bar{x}_i \rfloor \in \{0, \frac{1}{2}\}$. Therefore the extended formulation for $\text{conv}(X(G, b, I))$ (which will be introduced in Section 2) is compact. The main contribution of this paper is the explicit description of the projection of the polyhedron defined by this extended formulation in the original x-space.

The mixed-integer set $X(G, b, I)$ is related to certain mixed-integer sets that arise in the context of production planning (see [9]). The case when G is a star with center node in L and leaves in I has been studied by Pochet and Wolsey in [8], where they gave a compact extended formulation for the convex hull of feasible solutions. Günlük and Pochet [5] projected this formulation onto the original space, thus showing that the family of *mixing inequalities* gives the formulation in the x-space.

Miller and Wolsey [6] extended the results in [8] to general bipartite graphs, with the restriction that the partition (I, L) coincides with the bipartition (U, V) of the graph. Their result shows that the mixing inequalities associated with every single star of G having center a node in L and leaf nodes all nodes in I give a formulation for this case.

2 The Extended Formulation

We use here a modeling technique introduced by Pochet and Wolsey [8] and extensively investigated in [1].
Observation 2 allows to express each variable x_i , $i \in L$, as

$$x_i = \mu_i + \frac{1}{2}\delta_i,\ \mu_i \geq 0,\ 0 \leq \delta_i \leq 1,\ \mu_i, \delta_i \text{ integer.} \tag{7}$$

For now, we assume $I = \emptyset$, that is, $L = (U \cup V)$.

Lemma 3. *Let $ij \in E$, and suppose $x_i, x_j, \mu_i, \mu_j, \delta_i, \delta_j$ satisfy (7). If $b_{ij} = \frac{1}{2} \bmod 1$, x_i, x_j satisfy $x_i + x_j \geq b_{ij}$ if and only if*

$$\begin{aligned} \mu_i + \mu_j &\geq \lfloor b_{ij} \rfloor \\ \mu_i + \delta_i + \mu_j + \delta_j &\geq \lceil b_{ij} \rceil . \end{aligned} \tag{8}$$

If $b_{ij} = 0 \bmod 1$, x_i, x_j satisfy $x_i + x_j \geq b_{ij}$ if and only if

$$\begin{aligned} \mu_i + \delta_i + \mu_j &\geq b_{ij} \\ \mu_i + \mu_j + \delta_j &\geq b_{ij} . \end{aligned} \tag{9}$$

Proof: Assume $x_i, x_j, \mu_i, \mu_j, \delta_i, \delta_j$ satisfy (7). Then, if $b_{ij} = \frac{1}{2} \bmod 1$, constraint $x_i + x_j \geq b_{ij}$ is satisfied if and only if $\mu_i + \mu_j \geq \lfloor b_{ij} \rfloor$ and $\delta_i + \delta_j \geq 1$ whenever $\mu_i + \mu_j = \lfloor b_{ij} \rfloor$. If $b_{ij} = 0 \bmod 1$, the constraint is satisfied if and only if $\mu_i + \mu_j \geq b_{ij} - 1$ and $\delta_i = \delta_j = 1$ whenever $\mu_i + \mu_j = b_{ij} - 1$.
It is easy to see that these conditions are enforced by the above constraints. □

Observation 4. *Given $ij \in E$, the constraints (8) and (9) belong to the first Chvátal closure of the polyhedron defined by*

$$\begin{aligned} \mu_i + \frac{1}{2}\delta_i + \mu_j + \frac{1}{2}\delta_j &\geq b_{ij} \\ \mu_i, \mu_j &\geq 0 \\ \delta_i, \delta_j &\leq 1 \\ \delta_i, \delta_j &\geq 0 \end{aligned}$$

whenever $b_{ij} = \frac{1}{2} \bmod 1$ and $b_{ij} = 0 \bmod 1$, respectively.

By applying the unimodular transformation $\mu_i^0 = \mu_i$, $\mu_i^1 = \mu_i + \delta_i$, the constraints $x_i = \mu_i + \frac{1}{2}\delta_i,\ \mu_i \geq 0,\ 0 \leq \delta_i \leq 1$ become

$$x_i - \frac{1}{2}(\mu_i^0 + \mu_i^1) = 0 \tag{10}$$

$$\begin{aligned} \mu_i^0 &\geq 0 \\ 0 \leq \mu_i^1 - \mu_i^0 &\leq 1 \end{aligned} \tag{11}$$

and constraints (8) and (9) become:

$$\begin{aligned} \mu_i^0 + \mu_j^0 &\geq \lfloor b_{ij} \rfloor \\ \mu_i^1 + \mu_j^1 &\geq \lceil b_{ij} \rceil \end{aligned} \tag{12}$$

$$\begin{aligned} \mu_i^1 + \mu_j^0 &\geq b_{ij} \\ \mu_i^0 + \mu_j^1 &\geq b_{ij} \end{aligned} \tag{13}$$

Theorem 5. *The projection onto the space of the x variables of the polyhedron Q defined on the space of the variables (x, μ^0, μ^1) by the inequalities*

$$\begin{aligned} (10), (11) &\ \text{ for every } i \in U \cup V, \\ (12) &\ \text{ for every } ij \in E \text{ s.t. } b_{ij} = \tfrac{1}{2} \bmod 1 \\ (13) &\ \text{ for every } ij \in E \text{ s.t. } b_{ij} = 0 \bmod 1 \end{aligned}$$

is the polyhedron $\operatorname{conv}(X(G, b, \emptyset))$.

Proof: Since the variable x_i is determined by (10) for all $i \in U \cup V$, we only need to show that the polyhedron defined by inequalities (11) for every $i \in U \cup V$, (12) for every $ij \in E$ s.t. $b_{ij} = \frac{1}{2} \bmod 1$, and (13) for every $ij \in E$ s.t. $b_{ij} = 0 \bmod 1$, is integral. Let A_μ be the constraint matrix of the above system. Since G is a bipartite graph, then the matrix $\bar{A}$, obtained by multiplying by -1 the columns of A_μ relative to the variables μ_i^0, μ_i^1, $i \in V$, has at most a 1 and at most a -1 in each row. Therefore $\bar{A}$ is the transpose of a network matrix, so A_μ is totally unimodular (see [10]). Since the right-hand-sides of (11)-(13) are all integer, the statement follows from the theorem of Hoffman and Kruskal. □

Observation 6. *For any $i \in U \cup V$, x_i is integer valued if and only if $\delta_i = 0$. Therefore, for a given $I \subseteq (U \cup V)$, the polyhedron* $\operatorname{conv}(X(G, b, I))$ *is the projection onto the space of the x variables of the face Q_I of Q defined by the equations $\mu_1^1 - \mu_i^0 = 0,\ i \in I$ (which correspond to $\delta_i = 0,\ i \in I$).*

3 The Formulation in the Original Space

In this section we prove Theorem 1 by projecting the polyhedron Q_I onto the space of the x variables.

Let $p_i = \frac{\mu_i^0 - \mu_i^1}{2}$. The $\mu_i^0 = x_i + p_i$ and $\mu_i^1 = x_i - p_i$. The inequalities (10)-(13), defining Q, become:

$$\begin{array}{rll} p_i + p_j \geq \lfloor b_{ij} \rfloor - x_i - x_j, & ij \in E \text{ s.t. } b_{ij} = \frac{1}{2} \bmod 1, \\ -p_i - p_j \geq \lceil b_{ij} \rceil - x_i - x_j, & ij \in E \text{ s.t. } b_{ij} = \frac{1}{2} \bmod 1, \\ p_i - p_j \geq b_{ij} - x_i - x_j, & ij \in E \text{ s.t. } b_{ij} = 0 \bmod 1, \\ -p_i + p_j \geq b_{ij} - x_i - x_j, & ij \in E \text{ s.t. } b_{ij} = 0 \bmod 1, \\ p_i \geq -\frac{1}{2}, & i \in U \cup V, \\ -p_i \geq 0, & i \in U \cup V, \\ p_i \geq -x_i, & i \in U \cup V. \end{array}$$

By Observation 6, conv$(X(G, B, I))$ is the projection onto the x-space of the polyhedron defined by the above inequalities and by $p_i = 0$ for every $i \in I$.

Associate multipliers to the above constraints as follows:

$$\begin{array}{rr} (u_{ij}^{++}) & p_i + p_j \geq \lfloor b_{ij} \rfloor - x_i - x_j \\ (u_{ij}^{--}) & -p_i - p_j \geq \lceil b_{ij} \rceil - x_i - x_j \\ (u_{ij}^{+-}) & p_i - p_j \geq b_{ij} - x_i - x_j \\ (u_{ij}^{-+}) & -p_i + p_j \geq b_{ij} - x_i - x_j \\ (u_i^{\frac{1}{2}}) & p_i \geq -\frac{1}{2} \\ (u_i^0) & -p_i \geq 0 \\ (u_i^x) & p_i \geq -x_i \end{array} \tag{14}$$

Any valid inequality for conv$(X(G, b, I))$ has the form $\alpha_u x \geq \beta_u$, where

$$\alpha_u x = \sum_{b_{ij} = \frac{1}{2} \bmod 1} (u_{ij}^{++} + u_{ij}^{--})(x_i + x_j) + \sum_{b_{ij} = 0 \bmod 1} (u_{ij}^{+-} + u_{ij}^{-+})(x_i + x_j) + \sum_{i \in U \cup V} u_i^x x_i \tag{15}$$

$$\beta_u = \sum_{b_{ij} = \frac{1}{2} \bmod 1} (u_{ij}^{--} \lceil b_{ij} \rceil + u_{ij}^{++} \lfloor b_{ij} \rfloor) + \sum_{b_{ij} = 0 \bmod 1} (u_{ij}^{+-} + u_{ij}^{-+}) b_{ij} - \sum_{i \in L} \frac{1}{2} u_i^{\frac{1}{2}} \tag{16}$$

for some nonnegative vector $u = (u_{ij}^{++}, u_{ij}^{--}, u_{ij}^{+-}, u_{ij}^{-+}, u_i^{\frac{1}{2}}, u_i^0, u_i^x)$ such that $uP = 0$, where P is the column-submatrix of the above system (14) involving columns corresponding to variables p_i, $i \in L$ (see e.g. Theorem 4.10 in [7]). For instance the inequality $x_i + x_j \geq b_{ij}$, for $ij \in E$ with $b_{ij} = \frac{1}{2} \bmod 1$, is obtained by setting $u_{ij}^{++} = u_{ij}^{--} = \frac{1}{2}$, and all other entries of u to be 0.

We are interested in characterizing the nonnegative vectors u such that $uP = 0$ and $\alpha_u x \geq \beta_u$ is facet-defining for $\text{conv}(X(G, b, I))$, and such that the inequality $\alpha_u x \geq \beta_u$ is not of the form $x_i + x_j \geq b_{ij}$, for some $ij \in E$, or $x_i \geq 0$, for some $i \in U \cup V$. From now on we will assume, w.l.o.g., that the entires of u are integer and relatively prime.

We define an auxiliary graph $\Gamma_u = (L \cup \{d\}, F)$, where d is a dummy node not in $U \cup V$, and F is defined as follows.

- For every edge $ij \in E$ such that $i, j \in L$, there are $u_{ij}^{++} + u_{ij}^{--} + u_{ij}^{+-} + u_{ij}^{-+}$ parallel edges between i and j in F, each edge corresponding to a multiplier among $u_{ij}^{++}, u_{ij}^{--}, u_{ij}^{+-}, u_{ij}^{-+}$.
- For each node $i \in L$, there are $u_i^{\frac{1}{2}} + u_i^0 + u_i^x + \sum_{j \in I\,:\, ij \in E}(u_{ij}^{++} + u_{ij}^{--} + u_{ij}^{+-} + u_{ij}^{-+})$ parallel edges between d and i in F, each edge corresponding to a multiplier among $u_i^{\frac{1}{2}}$, u_i^0, u_i^x, or u_{ij}^{++}, u_{ij}^{--}, u_{ij}^{+-}, u_{ij}^{-+}, for some $j \in I$.

We impose a *bi-orientation* ω on Γ_u, that is, to each edge $e \in F$, and each endnode i of e that belongs to L, we associate the value $\omega(e, i) = \textit{tail}$ if e corresponds to an inequality of (14) where p_i has coefficient -1, while we associate the value $\omega(e, i) = \textit{head}$ if e corresponds to an inequality of (14) where p_i has coefficient $+1$. The dummy node d is neither a tail nor a head of any edge. Thus, each edge of Γ_u can have one head and one tail, two heads, two tails, or, if d is one of the two endnodes, only one head and no tail or only one tail and no head.

For each $i \in L$, we denote with $\delta_\omega^{in}(i)$ the number of edges in F of which i is a head, and with $\delta_w^{out}(i)$ the number of edges in F of which i is a tail.

We say that Γ_u is ω-eulerian if $\delta_\omega^{in}(i) = \delta_\omega^{out}(i)$ for every $i \in L$.

Observation 7. *Γ_u is ω-eulerian if and only if $uP = 0$.*

We define a *closed ω-eulerian walk* in Γ_u as a closed-walk in Γ_u,

$$v_0, e_0, v_1, e_1, \ldots, v_k, e_k, v_{k+1},$$

where $v_0 = v_{k+1}$, with the property that $\omega(e_{h-1}, v_h) \neq \omega(e_h, v_h)$ for every h such that v_h is in L, $h = 0, \ldots, k, k+1$, where the indices are taken modulo k. That is, if $v_h \in L$, then v_h is a head of e_{h-1} if and only if v_h is a tail of e_h.

Observation 8. *Γ_u is ω-eulerian if and only if Γ_u is the disjoint union of closed ω-eulerian walks. In particular, every node in $L \cup \{d\}$ has even degree in Γ_u.*

Observe that, if $v_0, e_0, \ldots, e_k, v_{k+1}$ is a closed ω-eulerian walk in Γ_u, then both graphs Γ', Γ'' on $L \cup \{d\}$ with edge-sets $F' = \{e_1, \ldots, e_k\}$ and $F'' = F \setminus F'$, respectively, are ω-eulerian. Suppose $F'' \neq \emptyset$. Then there are nonnegative integer vectors u' and u'', both different from zero, such that $u'P = 0$, $u''P = 0$, $\Gamma' = \Gamma_{u'}$ and $\Gamma'' = \Gamma_{u''}$, and $u = u' + u''$. By the fact that Γ' and Γ'' are ω-eulerian, and by the structure of the inequalities in (14), the vectors $(\alpha_{u'}, \beta_{u'})$ and $(\alpha_{u''}, \beta_{u''})$ are both non-zero. Furthermore $\alpha_u = \alpha_{u'} + \alpha_{u''}$ and $\beta_u = \beta_{u'} + \beta_{u''}$, contradicting the fact that $\alpha_u x \geq \beta_u$ is facet-defining and the entries of u are relatively prime.

Hence we have shown the following.

Observation 9. *Every closed ω-eulerian walk of Γ_u traverses all the edges in F. In particular, there exists a closed ω-eulerian walk $v_0, e_0, \ldots, e_k, v_{k+1}$ of Γ_u such that $F = \{e_h \mid h = 1, \ldots, k\}$.*

Suppose d has positive degree in Γ. Then we may assume, w.l.o.g., that $v_0 = v_{k+1} = d$. Suppose $d = v_h$ for some $h = 1, \ldots, k$. Then $v_0, e_0, v_1, \ldots, e_{h-1} v_h$ is a closed ω-eulerian walk, contradicting the previous observation. Hence we have the following.

Observation 10. *Node d has degree 0 or 2 in Γ_u.*

Next we show the following.

Lemma 11. *Every node in $L \cup \{d\}$ has degree 0 or 2 in Γ_u.*

Proof: We have already shown d has degree 0 or 2 in Γ_u. If d has degree 2, we assume $d = v_0 = v_{k+1}$, else v_0 is arbitrarily chosen. If there is a node in L with degree at least 4, then there exists distinct indices $s, t \in \{1, \ldots, k\}$ such that $v_s = v_t$. We choose s and t such that $t - s$ is positive and as small as possible. Therefore $C = v_s, e_s, \ldots, e_{t-1}, v_t$ is a cycle of Γ_u containing only nodes in L. Since G is a bipartite graph, C has even length, hence the edges in C can be partitioned into two matchings M_0, M_1 of cardinality $|C|/2$. We will denote with HH, TT, HT the sets of edges of F with, respectively, two heads, two tails, one head and one tail.

If v_s is the head of exactly one among e_s and e_{t-1}, then C is a closed ω-eulerian walk, contradicting Observation 9. Hence v_s is either a head of both e_s and e_{t-1} or a tail of both e_s and e_{t-1}. This shows that $|C \cap TT| = |C \cap HH| \pm 1$. Therefore there is an odd number of edges e in C such that $b_e = \frac{1}{2} \bmod 1$. By symmetry, we may assume $\sum_{e \in M_0} b_e \geq \sum_{e \in M_1} b_e + \frac{1}{2}$. Then the inequality

$$2 \sum_{i \in V(C)} x_i \geq \sum_{e \in C} b_e + \frac{1}{2} \tag{17}$$

is valid for $\mathrm{conv}(X(G, b, I))$, since it is implied by the valid inequalities $x_i + x_j \geq b_{ij}$, $ij \in M_0$, because

$$2 \sum_{i \in V(C)} x_i = 2 \sum_{ij \in M_0} (x_i + x_j) \geq 2 \sum_{ij \in M_0} b_{ij} \geq \sum_{e \in M_0} b_e + \sum_{e \in M_1} b_e + \frac{1}{2} = \sum_{e \in C} b_e + \frac{1}{2}.$$

Case 1: Node v_s is a tail of both e_s and e_{t-1}.

Then $|C \cap TT| = |C \cap HH| + 1$, hence

$$\sum_{e \in C \cap TT} \lfloor b_e \rfloor + \sum_{e \in C \cap HH} \lceil b_e \rceil + \sum_{e \in C \cap HT} b_e = \sum_{e \in C} b_e + \frac{1}{2}. \tag{18}$$

Let u' be the vector obtained from u as follows

$$\begin{cases} u'^{**}_{ij} = u^{**}_{ij} - 1 \text{ for every } ij \in C \\ u'^{0}_{v_s} = u^{0}_{v_s} + 2 \end{cases}$$

all other components of u' and u being identical, where u_{ij}^{**} is the variable among $u_{ij}^{++}, u_{ij}^{--}, u_{ij}^{+-}, u_{ij}^{-+}$ corresponding to edge ij of C.

Then one can easily see that $\Gamma_{u'}$ is the graph obtained from Γ_u by removing the edges $e_s, \ldots, e_t$, and adding two parallel edges $v_s d$ both with tail in v_s, hence $\Gamma_{u'}$ is ω-eulerian and $u'P = 0$. By (18)

$$\beta_{u'} = \beta_u - \sum_{e \in C} b_e - \frac{1}{2},$$

while by construction

$$\alpha_u x = \alpha_{u'} x + 2 \sum_{i \in V(C)} x_i.$$

Thus $\alpha_u x \geq \beta_u$ can be obtained by taking the sum of $\alpha_{u'} x \geq \beta_{u'}$ and (17), contradicting the assumption that $\alpha_u x \geq \beta_u$ is facet-defining.

Case 2: Node v_s is a head of both e_s and e_{t-1}.

Then $|C \cap TT| = |C \cap HH| - 1$, hence

$$\sum_{e \in C \cap TT} \lfloor b_e \rfloor + \sum_{e \in C \cap HH} \lceil b_e \rceil + \sum_{e \in C \cap HT} b_e = \sum_{e \in C} b_e - \frac{1}{2}. \tag{19}$$

Let u' be the vector obtained from u as follows

$$\begin{cases} u'^{**}_{ij} = u^{**}_{ij} - 1 \text{ for every } ij \in C \\ u'^{\frac{1}{2}}_{v_s} = u^{\frac{1}{2}}_{v_s} + 2 \end{cases}$$

all other components of u' and u being identical.
Then one can easily see that $\Gamma_{u'}$ is the graph obtained from Γ_u by removing the edges $e_s, \ldots, e_t$, and adding two parallel edges $v_s d$ both with head in v_s, hence $u'P = 0$. By (19)

$$\beta_{u'} = \beta_u - \sum_{e \in C} b_e + \frac{1}{2} - 2\frac{1}{2},$$

while by construction

$$\alpha_u x = \alpha_{u'} x + 2 \sum_{i \in V(C)} x_i.$$

Thus $\alpha_u x \geq \beta_u$ can be obtained by taking the sum of $\alpha_{u'} x \geq \beta_{u'}$ and (17), contradicting the assumption that $\alpha_u x \geq \beta_u$ is facet-defining.

□

We are now ready to give the proof of the main theorem.

Proof of Theorem 1. We show that all facet-defining inequalities $\alpha_u x \geq \beta_u$, where u is nonnegative, integral, and with entries that are relatively prime, that are not inequalities in (2) or (4), are of the form (3).

First we show the following.

$$\sum_{ij\in E} u_{ij}^{--} > \sum_{ij\in E} u_{ij}^{++} + \sum_{i\in U\cup V} u_i^{\frac{1}{2}} \tag{20}$$

In fact, we can write the inequality

$$\alpha_u x \geq \sum_{b_{ij}=\frac{1}{2} \bmod 1} (u_{ij}^{--} + u_{ij}^{++}) b_{ij} + \sum_{b_{ij}=0 \bmod 1} (u_{ij}^{+-} + u_{ij}^{-+}) b_{ij}$$

as nonnegative combination of inequalities of the form (2) or (4), therefore we must have

$$\beta_u > \sum_{b_{ij}=\frac{1}{2} \bmod 1} (u_{ij}^{--} + u_{ij}^{++}) b_{ij} + \sum_{b_{ij}=0 \bmod 1} (u_{ij}^{+-} + u_{ij}^{-+}) b_{ij}.$$

Thus

$$\begin{aligned} 0 &< \beta_u - \sum_{b_{ij}=\frac{1}{2} \bmod 1} (u_{ij}^{--} + u_{ij}^{++}) b_{ij} - \sum_{b_{ij}=0 \bmod 1} (u_{ij}^{+-} + u_{ij}^{-+}) b_{ij} \\ &= \frac{1}{2} (\sum_{ij\in E} u_{ij}^{--} - \sum_{ij\in E} u_{ij}^{++} - \sum_{i\in U\cup V} u_i^{\frac{1}{2}}) \end{aligned}$$

which proves (20).

By Lemma (11) and Observation (9), Γ_u consists of an induced cycle C and isolated nodes, where every node in $V(C) \cap L$ is a head of exactly one edge and a tail of exactly one edge.

If d is an isolated node, then each edge ij of C corresponds to a variable of the form u_{ij}^{**}, and since the total number of heads in C equals the number of tails, then $\sum_{ij\in E} u_{ij}^{--} = \sum_{ij\in E} u_{ij}^{++}$ and $\sum_{i\in U\cup V} u_i^{\frac{1}{2}} = 0$, contradicting (20). Thus we may assume that $C = v_0, e_0, \ldots, e_k, v_{k+1}$ where $d = v_0 = v_{k+1}$.

Claim: The following are the only possible cases, up to symmetry.
1. Edges dv_1, dv_k of Γ_u correspond to variables $u_{v_1}^x$ and $u_{v_k}^x$, respectively;
2. dv_1 corresponds to variable $u_{wv_1}^{--}$ or $u_{wv_1}^{-+}$ for some $w \in I$, and dv_k corresponds to $u_{v_k}^x$;
3. dv_1 corresponds to variables $u_{wv_1}^{--}$ or $u_{wv_1}^{-+}$ for some $w \in I$, and dv_k corresponds to variable $u_{w'v_k}^{--}$ or $u_{w'v_k}^{-+}$ for some $w' \in I$.

Proof of claim. If v_1 is a head of e_0 and v_k is a head of e_k, then the number of edges among $e_1, \ldots, e_{k-1}$ with two tails is one plus the number of edges with two heads. Since the former correspond to variables of type u_{ij}^{--} for some $ij \in E$, and the latter correspond to to variables of type u_{ij}^{++} for some $ij \in E$, then by (20) dv_1 does not correspond to variable $u_{v_1}^{\frac{1}{2}}$ or to a variable $u_{wv_1}^{++}$ for any $w \in I$, and dv_k does not correspond to variable $u_{v_k}^{\frac{1}{2}}$ or to a variable $u_{wv_k}^{++}$ for any $w \in I$, thus one of the above three cases holds.

If v_1 is a tail of e_0 and v_k is a head of e_k, then the number of edges among $e_1, \ldots, e_{k-1}$ with two tails is equal the number of edges with two heads. By (20), dv_1 corresponds to variable $u^{--}_{wv_1}$ for some $w \in I$, and dv_k corresponds to either $u^x_{v_k}$ or to a variable $u^{-+}_{w'v_k}$ for some $w' \in I$, thus case 2 or 3 holds.

If v_1 is a tail of e_0 and v_k is a tail of e_k, then the number of edges among $e_1, \ldots, e_{k-1}$ with two tails is equal one minus the number of edges with two heads. By (20), dv_1 corresponds to variable $u^{--}_{wv_1}$ for some $w \in I$, and dv_k corresponds to a variable $u^{--}_{w'v_k}$ for some $w' \in I$, thus case 3 holds. This completes the proof of the claim.

Case 1: Edges dv_1, dv_k of Γ_u correspond to variables $u^x_{v_1}$ and $u^x_{v_k}$, respectively.

In this case the path $P = v_1, e_1, \ldots, e_{k-1}, v_k$ of Γ_u is also a path of G containing only nodes in L, and P contains an odd number of edges e such that $b_e = \frac{1}{2} \bmod 1$. The inequality $\alpha_u x \geq \beta_u$ is then $2x(V(P)) \geq b(P) + \frac{1}{2}$. The edges of P can be partitioned into two matchings M_0 and M_1, thus we may assume, w.l.o.g., $\sum_{e \in M_0} b_e \geq \sum_{e \in M_1} b_e + \frac{1}{2}$. Thus $2x(V(P)) \geq 2\sum_{ij \in M_0}(x_i + x_j) \geq 2\sum_{ij \in M_0} b_{ij} \geq \sum_{e \in M_0} b_e + \sum_{e \in M_1} b_e + \frac{1}{2} = b(P) + \frac{1}{2}$, hence $\alpha_u x \geq \beta_u$ is not facet-defining.

Case 2: dv_1 corresponds to variable $u^{--}_{wv_1}$ or $u^{-+}_{wv_1}$ for some $w \in I$, and dv_k corresponds to $u^x_{v_k}$.

In this case, $P = w, v_1, e_1, \ldots, e_{k-1}, v_k$ is an odd I-path of G between $w \in I$ and $v_k \in L$. The inequality $\alpha_u x \geq \beta_u$ is $2x(V(P) \cap L) + x_w \geq b(P) + \frac{1}{2}$, which is one of the inequalities in (3).

Case 3: dv_1 corresponds to variables $u^{--}_{wv_1}$ or $u^{-+}_{wv_1}$ for some $w \in I$, and dv_k corresponds to variable $u^{--}_{w'v_k}$ or $u^{-+}_{w'v_k}$ for some $w' \in I$.

If $w \neq w'$, then the path $P = w, v_1, e_1, \ldots, e_{k-1}, v_k, w'$ is an odd I-path of G between $w \in I$ and $w' \in I$. The inequality $\alpha_u x \geq \beta_u$ is $2x(V(P) \cap L) + x_w + x_{w'} \geq b(P) + \frac{1}{2}$, which is one of the inequalities in (3).

If $w = w'$, then we must have $v_1 \neq v_k$, since otherwise v_1 would be either the head or the tail of both edges of Γ_u incident to v_1. Thus $C' = w, v_1, \ldots, v_k, w$ is a cycle of G. Since G is a bipartite graph, C' has even length, hence the edges in C' can be partitioned into two matchings M_0, M_1 of cardinality $|C'|/2$. Since C' contains an odd number of edges e such that $b_w = \frac{1}{2} \bmod 1$, then we may assume, w.l.o.g., $\sum_{e \in M_0} b_e \geq \sum_{e \in M_1} b_e + \frac{1}{2}$. The inequality $\alpha_u x \geq \beta_u$ is $2x(V(C')) \geq b(C') + \frac{1}{2}$. But $2x(V(C')) = 2\sum_{ij \in M_0}(x_i + x_j) \geq 2\sum_{ij \in M_0} b_{ij} \geq \sum_{e \in M_0} b_e + \sum_{e \in M_1} b_e + \frac{1}{2} = b(C') + \frac{1}{2}$, hence $\alpha_u x \geq \beta_u$ is not facet-defining. □

4 Separation

Theorem 5 and Observation 6 imply that the problem of minimizing a linear function over the set $X(G, b, I)$ is solvable in polynomial time, since it reduces to solving a linear programming problem over the set of feasible points for (10)-(13).

In this section we give a combinatorial polynomial-time algorithm for the separation problem for the set $\mathrm{conv}(X(G,b,I))$, thus giving an alternative proof that the problem of optimizing a linear function over such polyhedron, and thus over $X(G,b,I)$, is polynomial.

Clearly, given a nonnegative vector x^*, we can check in polynomial-time whether x^* satisfies (2) for every edge. Thus, by Theorem 1, we only need to describe a polynomial-time algorithm that, given a nonnegative vector x^* satisfying (2), either returns an inequality of type (3) violated by x^*, or proves that none exists.

For every $ij \in E$, let $s^*_{ij} = x^*_i + x^*_j - b_{ij}$. Since x^* satisfies (2), then s^*_e is nonnegative for every $e \in E$. Let $P = v_1, \dots v_n$ be an odd I-path.

Claim. The vector x^* satisfies $2x^*(V(P)\cap L) + x^*(V(P)\cap I) \geq b(P) + \frac{1}{2}$ if and only if $s^*(P) + x^*(\{v_1, v_n\}\cap L) \geq \frac{1}{2}$.

Indeed, assume $v_1 \in I$. If $v_n \in I$ then

$$\sum_{i=1}^{n-1} s^*_{v_i v_{i+1}} = \sum_{i=1}^{n-1}(x^*_{v_i} + x^*_{v_{i+1}} - b_{v_i v_{i+1}})$$

gives the equality $s^*(P) = 2x^*(V(P)\cap L) + x^*(V(P)\cap I) - b(P)$, hence $2x^*(V(P)\cap L) + x^*(V(P)\cap I) \geq b(P) + \frac{1}{2}$ if and only if $s^*(P) \geq \frac{1}{2}$.

If $v_n \notin I$, then

$$\sum_{i=1}^{n-1} s^*_{v_i v_{i+1}} + x^*_{v_n} = \sum_{i=1}^{n-1}(x^*_{v_i} + x^*_{v_{i+1}} - b_{v_i v_{i+1}}) + x^*_{v_n}$$

gives the equality $s^*(P) + x^*_{v_n} = 2x^*(V(P)\cap L) + x^*(V(P)\cap I) - b(P)$, hence $2x^*(V(P)\cap L) + x^*(V(P)\cap I) \geq b(P) + \frac{1}{2}$ if and only if $s^*(P) + x^*_{v_n} \geq \frac{1}{2}$.

This completes the proof of the Claim.

Therefore, if we assign length s^*_e to every $e \in E$, we need to give an algorithm that, for any two nodes r, t such that $r \in I$, either determines that the shortest odd I-path between r and t (if any) has length at least $\frac{1}{2} - x^*(\{t\}\cap L)$, or returns an odd I-path P for which $2x^*(V(P)\cap L) + x^*(V(P)\cap I) < b(P) + \frac{1}{2}$.

Observe that any walk W between r and t that contains an odd number of edges e such that $b_e = \frac{1}{2} \bmod 1$ either contains a sub-path P that is an odd I-path or it contains a cycle C that contains an odd number of edges e such that $b_e = \frac{1}{2} \bmod 1$. In the former case, either both endnodes of P are in I, or t is the only endnode of P in L. Hence, if $s^*(W) < \frac{1}{2} - x^*(\{t\}\cap L)$, then also $s^*(P) < \frac{1}{2} - x^*(\{t\}\cap L)$, hence $2x^*(V(P)\cap L) + x^*(V(P)\cap I) < b(P) + \frac{1}{2}$. In the second case, since G is bipartite, the edges of C can be partitioned into two matchings M_0 and M_1 such that $b(M_0) \geq b(M_1) + \frac{1}{2}$. Thus $s^*(C) = \sum_{ij\in C}(x^*_i + x^*_j - b_{ij}) = 2x^*(V(C)) - b(C) \geq 2(x^*(V(C)) - b(M_0)) + \frac{1}{2} = 2\sum_{ij\in M_0}(x^*_i + x^*_j - b_{ij}) + \frac{1}{2} \geq \frac{1}{2}$, hence $s^*(W) \geq \frac{1}{2}$.

Thus we only need to find, for every pair $r, t \in U \cup V$ with $r \in I$, the shortest walk W between r and t, w.r.t. the distance s^*, among all such walks containing an odd number of edges e such that $b_e = \frac{1}{2} \bmod 1$. If, for a given choice of r, t,

$s(W) < \frac{1}{2} - x^*(\{t\} \cap L)$, then by the above argument we can find in polynomial time a sub-path P of W such that P is an odd I-path and $2x^*(V(P) \cap L) + x^*(V(P) \cap I) < b(P) + \frac{1}{2}$, otherwise we can conclude that $x^* \in \text{conv}(X(G, b, I))$.

To conclude, we only need to show a polynomial-time algorithm that, given an undirected graph Γ with nonnegative lengths on the edges ℓ_e, $e \in E(\Gamma)$, a subset $F \subseteq E(\Gamma)$, and a pair of nodes $r, t \in V(\Gamma)$, determines the walk W of minimum length between r and t such that $|E(W) \cap F|$ is odd, or determines that no such walk exists. The latter problem can be solved in polynomial time. Since, as far as we know, this fact is folklore, we briefly describe an algorithm.

We construct a new graph Γ' as follows. For every node $v \in V(\Gamma)$, there is a pair of nodes v, v' in $V(\Gamma')$. For every edge $uv \in E(\Gamma)$, $E(\Gamma')$ contains the edges uv' and $u'v$ if $uv \in F$, and the edges uv and $u'v'$ if $uv \notin F$, each with length ℓ_{uv}. One can verify that a walk W between r and t with an odd number of edges in F exists in Γ if and only if there exists a walk of the same length between r and t' in Γ'. Hence we only need to find a shortest path between r and t' in Γ', if any exists, and output the corresponding walk in Γ.

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